DUSO Mathematics League 2016 - 2017

CONTEST #2.

SOLUTIONS

2 - 1. $\boxed{5}$ Let B be the number of boys and G be the number of girls. The second sentence in the problem implies $B - 1 = G$. The third sentence in the problem implies $B = 3(G - 1)$. Combining these equations gives the equation $3(G - 1) - 1 = G$, which solves to obtain $3G - 4 = G \to 2G = 4 \to G = 2 \to B = 3$. There are $2 + 3 = \mathbf{5}$ children in the family.

2 - 2. $\boxed{40}$ Factor to obtain $x^2y - xy^2 = xy(x - y) = 5(8) = \mathbf{40}$. Note that this solution does not require finding the values of either x or y.

---

2 - 3. $\boxed{0}$ The equation of the line can be re-written as $5(4x + 3y) = 2016$. This implies that if x and y are integers, then 5 divides 2016. Since this is never true, $N = \mathbf{0}$.

2 - 4. $\boxed{232\pi}$ The cross section is a circle of radius $\sqrt{9} = 3$. The surface area of the sphere is $S = 4\pi r^2$ where $r^2 = 7^2 + 3^2 = 58$, so $S = \mathbf{232}\pi$.

---

2 - 5. $\boxed{1}$ Notice that $2^3 + 5 = 13$, so 13 is a prime that satisfies the conditions of the problem. Any other prime is an odd number, and odd numbers have odd cubes, and adding 5 to an odd cube makes the result even (and therefore not prime). Therefore, there is **1** such prime.

2 - 6. $\boxed{(8,5)}$ Substituting, $\dfrac{x + \frac{1}{x-3}}{x - 3 + \frac{1}{x}} = \dfrac{8}{5}$. This implies $5x + \dfrac{5}{x-3} = 8x - 24 + \dfrac{8}{x}$. Multiplying through by $x(x - 3)$ gives $5x^2(x - 3) + 5x = 8x^2(x - 3) - 24x(x - 3) + 8x - 24$, which implies $0 = 3x^3 - 33x^2 + 75x - 24 \to 0 = x^3 - 11x^2 + 25x - 8$. Because the value of x is an integer, x must be a factor of 8. Substitution shows that $x = 1$, $x = 2$, and $x = 4$ are not solutions, but $x = 8$ is, so $x = 8 \to y = 5$. The ordered pair is $(\mathbf{8}, \mathbf{5})$.

Alternate Solution: Clear the denominators in the complex fraction: $\dfrac{x + \dfrac{1}{y}}{y + \dfrac{1}{x}} = \dfrac{xy + 1}{y} \cdot \dfrac{x}{xy + 1}$, so the complex fraction reduces to $\dfrac{x}{y}$, so $\dfrac{x}{y} = \dfrac{8}{5}$ with $x - y = 3$, so $(x, y) = (8, 5)$.

Author: George Reuter - coachreu@gmail.com – Reviewer: Michael Curry - currymath@gmail.com

T-1. Compute the number of positive integer factors of 2016.
T-1Sol. $\boxed{36}$ Because $2016 = 2^5 \cdot 3^2 \cdot 7$, the number of positive integer factors is $(5+1)\cdot(2+1)\cdot(1+1) = \mathbf{36}$.

T-2. Given rectangle $ABCD$ and point P in the interior of $ABCD$. If $PA = 7$, $PB = 15$, and $PC = 24$, compute PD.
T-2Sol. $\boxed{\mathbf{20}}$ For any interior point P of a rectangle $ABCD$, $PA^2 + PC^2 = PB^2 + PD^2$. Try to prove it! By substituting, $PB^2 = 7^2 + 24^2 - 15^2 \rightarrow PB = \mathbf{20}$.

T-3. In a box are 2 red stickers, 4 white stickers, and 6 blue stickers. Six stickers are chosen at random without replacement. Compute the probability that the six are 1 red, 2 white, and 3 blue.
T-3Sol. $\boxed{\mathbf{\frac{20}{77}}}$ Choose 1 of the 2 reds, 2 of the 4 whites, and 3 of the 6 blues. Thus, the probability is $\dfrac{\binom{2}{1}\cdot\binom{4}{2}\cdot\binom{6}{3}}{\binom{12}{6}}$. This is equivalent to $\dfrac{2\cdot\frac{4\cdot3}{2\cdot1}\cdot\frac{6\cdot5\cdot4}{3\cdot2\cdot1}}{\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7}{6\cdot5\cdot4\cdot3\cdot2\cdot1}}$, or $\mathbf{\dfrac{20}{77}}$.

Author: George Reuter - coachreu@gmail.com – Reviewer: Michael Curry - currymath@gmail.com